A THEOREM ON THE STRUCTURE OF CELL-DECOMPOSITIONS OF ORIENTABLE 2-MANIFOLDS

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1. *Introduction.* For a cell compex M decomposing the closed orientable 2-manifold P_g of genus g let $p_i(M)$ and $v_j(M)$ denote the number of i-gonal cells (faces, countries) and j-valent vertices of the graph (= 1 − skeleton) of M, respectively. It will be supposed that $i, j \geqslant 3$. From Euler's formula follows

$$\sum_{k \geqslant 3} (4-k)(p_k(M) + v_k(M)) = 8(1-g),$$

and

$$\sum_{k \geqslant 3} k p_k(M) = \sum_{k \geqslant 3} k v_k(M) = 2e$$

is even, where e is the number of edges of M. As seen, the relation above does not impose restrictions on the numbers $p_4(M)$, $v_4(M)$. In an attempt to characterize the vectors $\{p_i(M)\}$, $\{v_i(M)\}$ which partially determine the combinatorial structure of the cell-decompositions, the first step could involve answering the question: Given sequences $p = (p_3, p_5, \ldots)$, $v = (v_3, v_5, \ldots)$ of non-negative integers satisfying conditions

$$\left.\begin{aligned} &\sum_{k \geqslant 3} (4-k)(p_k + v_k) = 8(1-g), \\ \text{and} \quad &\sum_{k \geqslant 3} k p_k \equiv 0 \pmod 2, \end{aligned}\right\} \qquad (1)$$

does there exist a cell-decomposition M of P_g, for which $p_i(M) = p_i$, $v_j(M) = v_j$ for all $i, j \neq 4$? (If so, the sequences p, v are called *realizable* in the sequel. M itself is a *realization* of p, v. For brevity's sake we will often use the word *map* instead of cell-decomposition in the sequel.)

The problem was formulated and, for $g = 0$, solved in [3] by B. Grünbaum, who gave friendly encouragement to our further investigations. For $g = 1$ the solution is given in D. Barnette–E. Jucovič–M. Trenkler [1] (cf. J. Zaks [5] for a special case). For $g \geqslant 3$, assuming $v = (0, 0, 0 \ldots)$, E. Jucovič–M. Trenkler [5] gives the answer. The aim of the present paper is to settle the entire problem by proving the following

MAIN THEOREM. *Given sequences of non-negative integers* $p = (p_3, p_5, \ldots)$, $v = (v_3, v_5, \ldots)$ *satisfying, for some integer g, the conditions*

$$\sum_{k \geqslant 3} (4-k)(p_k + v_k) = 8(1-g)$$

and

$$\sum_{k \geqslant 3} k p_k \equiv 0 \pmod 2,$$

there exists a cell-decomposition M of the closed orientable 2-manifold P_g *of genus g with* $p_i(M) = p_i$, $v_j(M) = v_j$ *for all* $i, j \neq 4$, *except in the cases* $g = 1$ *and* $p = (1, 1, 0, 0, \ldots)$, $v = (0, 0, \ldots)$ *or* $p = (0, 0, \ldots)$, $v = (1, 1, 0, 0, \ldots)$.

Because of the results in [1, 3, 4] it suffices to prove the Main theorem for $g \geqslant 2$ This will be done in two theorems.

[MATHEMATIKA 20 (1973), 63–82]

2. THEOREM 1. *Every pair of sequences p, v satisfying* (1) *with $g = 2$ is realizable on P_2.*

The proof consists of constructing, for every pair of sequences p, v satisfying (1), its realization. Unfortunately, many cases and subcases must be considered. In most of them first a toroidal map is constructed containing all the required k-gonal faces and m-valent vertices, $k, m \neq 4$, an unspecified number of quadrangles and 4-valent vertices, and two polygons O_1, O_2 with the same number of vertices which are all 3-valent. These polygons O_1, O_2, called "*openings*" in the sequel, are joined by a handle in the concluding stage of the construction; the handle is a cylindrical map containing quadrangles and 4-valent vertices, except those on the boundaries which are 3-valent. In some cases we shall construct two toroidal maps containing one opening each. One of the toroidal maps will realize the required faces (*face-part* of the map), the second one will realize the required vertices (*vertex-part*). The last step of the construction consists of unifying the two toroidal maps by identifying vertices and edges of the two openings. From the described strategy of the construction is seen that it is very important to ensure that the openings have equal numbers of 3-valent vertices.

Case I. $\sum_{k \geqslant 5} (k-4) p_k \geqslant 10$. In this case it follows from (1) that $p_3 \geqslant 2$ or $v_3 \geqslant 2$.

On each of two parallel sides of a rectangle R assign

$$\left[\tfrac{1}{2} \sum_{k \geqslant 5} (k-4) p_k\right] + \left[\tfrac{1}{2} \sum_{k \geqslant 5} (k-4) v_k\right] + 1 = s + 1$$

points $K_1, \ldots, K_s, K_{s+1}$, on the mid-line parallel to these sides assign points $L_1, \ldots, L_s, L_{s+1}$. Decompose one half of R into two hexagons O_1, O_2 and $2s-5$ quadrangles and $2s-8$ triangles as depicted by full lines in Fig. 1. In the sequel, this submap of the map constructed will be called the *a-part*, and the other submap will be called the *b-part*. To form a torus, equally marked vertices are identified, as well as the pairs K_1, K_{s+1}, and L_1, L_{s+1}. The same is done with pairs of points appearing in the course of the following construction on the boundary of R. So not only equal numbers of trivalent vertices on the openings but also equal numbers of vertices on opposite sides of R must be maintained. The paths $K_1 L_1 \ldots$ and $K_{s+1} L_{s+1} \ldots$ will be called "*borders*" of R in the sequel.

Before forming the torus the required k-gons ($k \geqslant 5$) must be cut off from the polygon $L_1 \ldots L_s K_s \ldots K_1$. If $k = 2m$, join a point P_1 between K_{m-1} and K_m with a point P_2 between L_{m-1} and L_m by an arc in the b-part. If P_2 does not belong to an edge of an opening the arc $P_1 P_2$ is extended by an edge $P_2 P_1$ into the a-part. If this is not the case one of the openings has increased its number of vertices by two; it will be shown later how balancing up the number of vertices of the second opening is performed.

The faces with odd numbers of vertices are cut off in pairs. Suppose we need a k-gon, $k = 2m+1$, and a h-gon, $h = 2n+1$, and these are the first we are constructing. Join, in the b-part, a point Q_1 between K_{m-1} and K_m with a point Q_4 between L_m and L_{m+1}, then join a point Q_2 between K_m and K_{m+1} with a point Q_3 between L_{m-1} and L_m. A point P_1 between K_{n+m-2} and K_{n+m-1} must now be joined with a point P_2 between L_{n+m-2} and L_{n+m-1} (dashed lines in Fig. 1, where $k = 9$, $h = 7$). Again, as before, it is necessary to balance up the number of vertices on the openings. Another complication arises when an odd number of faces having $\geqslant 5$

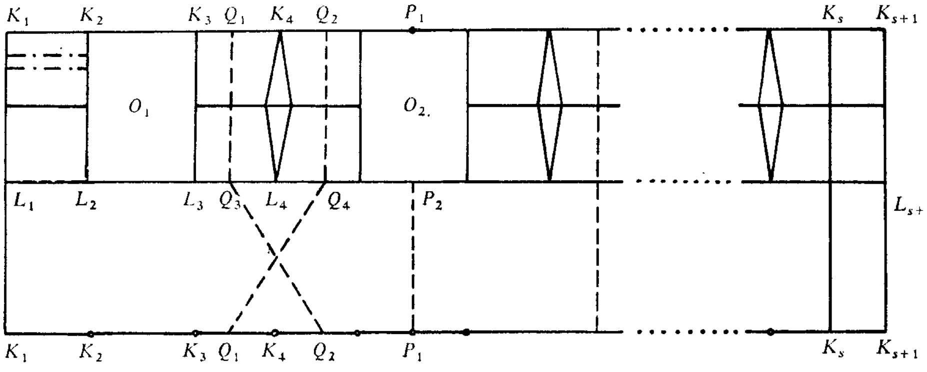

FIG. 1

edges is required; from (1) it follows that p_3 must be odd in this case. Let the last face we are constructing in the b-part have an odd number $k \geqslant 7$ of edges. We cut off a $(k - 1)$-gon F as before. If F is adjacent to an opening O_i then O_i is surrounded by a belt consisting of 7 quadrangles and one triangle as depicted in Fig. 2a. If F is not adjacent to an opening, i.e. a double triangle meets it, this double triangle is replaced by a triple of triangles and a quadrangle as depicted in Fig. 2b. Both these procedures increase the number of vertices of the face F by one. If $k = 5$ the triangle in the belt mentioned above is put next to an opening, not meeting the b-part (Fig. 2c).

The procedure of balancing the number of vertices on the opening O_1 or O_2, if this is necessary, will now be described. If, for example, on O_2 $c = 2d$ (c is always even!) trivalent vertices must be added, join d points between L_5 and L_6 with d points between K_3 and K_4 and these d points between K_3 and K_4 with d points between K_5 and K_6 by non-intersecting arcs. These form in the b-part as well as in the a-part quadrangles only (Fig. 3). This procedure of increasing the number of vertices of an opening is made possible since the double triangle can be placed between O_1

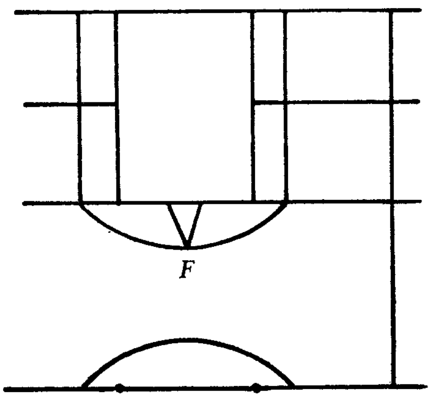

FIG. 2a

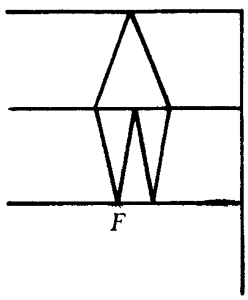

FIG. 2b

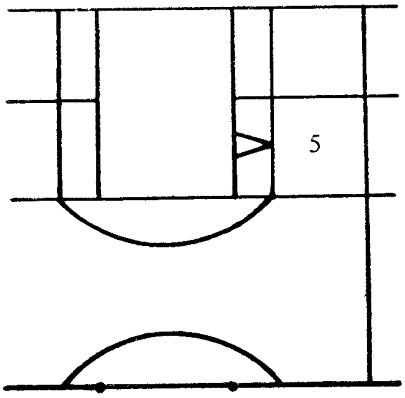

FIG. 2c

and O_2 when $p_3 \geqslant 2$ (Fig. 1). If $p_3 < 2$, and so $v_3 \geqslant 2$, the double triangle is replaced by the pair of trivalent vertices K_4, L_4 as shown in Fig. 5, and the balancing proceeds similarly.

The above argument settles subcase

(a) $v = (0, 0, \ldots)$.

Now consider the subcase

(b) $\sum_{k \geqslant 3} (4-k) v_k = 0, \ \sum_{i \neq 4} v_i \neq 0, \ v \neq (1, 1, 0, \ldots)$.

From the assumption made in case I it follows that after realizing the face-part a part R_v of the map R is available which does not contain an opening (redrawn in full lines in Fig. 4). In R_v we cut off v_i i-gonal faces for all $i \geqslant 5$ as before (the submap R_v thus obtained will be called the dual of the vertex-part in the sequel). The numbers of vertices on the openings are balanced, and then R_v is dualized (the dual of the map drawn in full lines in Fig. 4 is indicated by dashed lines). Joining both the face-part and the vertex-part is straightforward. (The pairs of vertices k_{s+1} and K_1, l_{s+1} and L_1 are identified, etc.)

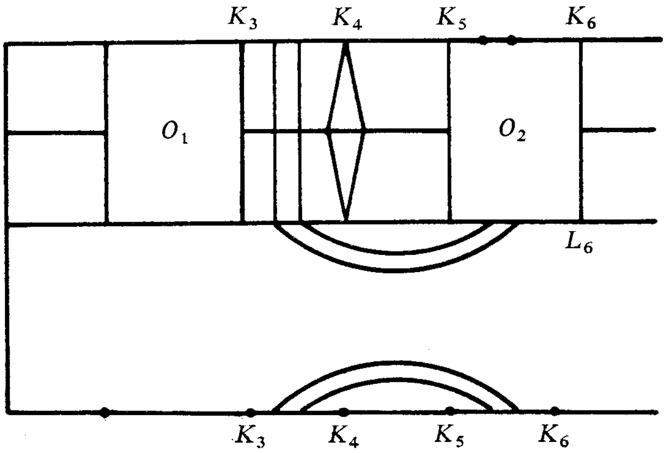

FIG. 3

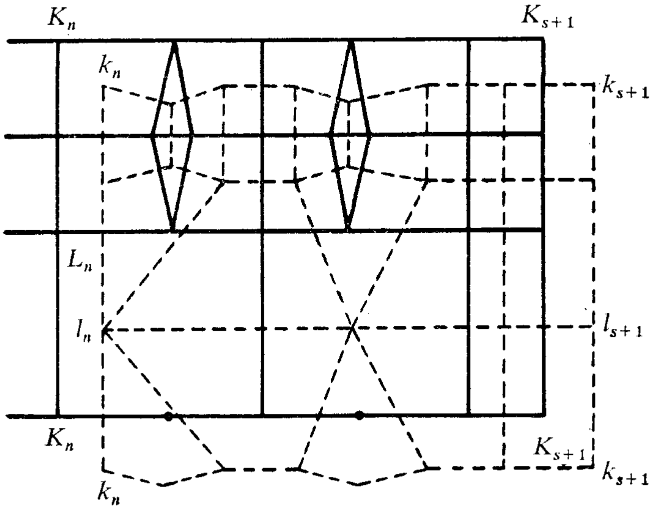

FIG. 4

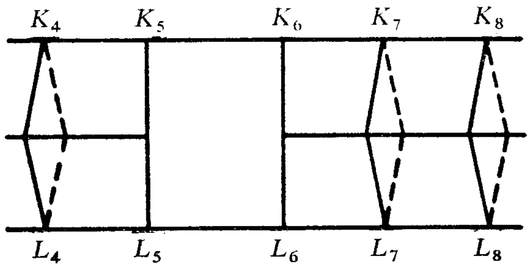

FIG. 5

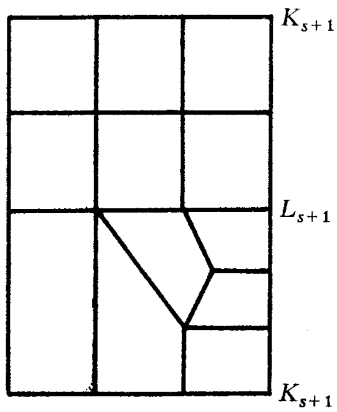

FIG. 6

(c) $\sum_{k\geqslant 3}(4-k)v_k > 0$, $\sum_{i\neq 4} v_i \neq 0$, $v \neq (1,1,0,0,\ldots)$. In the face-part $\sum_{k\geqslant 3}(4-k)v_k = 2w$ triangles are changed into w pairs of trivalent vertices by removing a pair of edges from each of the w double triangles. (See Fig. 5 where the removed edges are marked by dashed lines.)

(d) $\sum_{k\geqslant 3}(4-k)v_k < 0$, $\sum_{i\neq 4} v_i \neq 0$, $v \neq (1,1,0,\ldots)$. In the dual of the vertex-part $\sum_{k\geqslant 3}(k-4)v_k = p_3' \equiv O \pmod 2$ triangles are changed into trivalent vertices by the procedure above, and then R_v is dualized.

Remark. The processes described above of changing triangles into trivalent vertices and vice versa will be often used in the sequel without stating it explicitly.

(e) $v = (1,1,0,\ldots)$. The vertex-part is depicted in Fig. 6 and is added to the face-part where it is necessary to increase the number of vertices on the border $K_1 L_1$ by two. This is performed as depicted in Fig. 1 by dot-and-dash lines, i.e. two points between K_1 and L_1 are joined with two points between K_2 and L_2 by non-intersecting edges. Then the number of vertices of O_2 is increased by two as before (Fig. 3).

Case II. $\sum\limits_{k\geqslant 5}(k-4)p_k = 8$ or 9, $p_5 + 2p_6 \geqslant 2$, $\sum\limits_{k\geqslant 5}(k-4)v_k < 4$.

In this case the map in Fig. 1 cannot be used because there exist pairs of sequences satisfying (1) and the above conditions with $p_3 < 2$ and $v_3 < 2$. The double triangle between O_1 and O_2 would be superfluous. Moreover in cutting off the last k-gon, $k \geqslant 5$, we should be forced to dissect the opening O_2, which makes it impossible to continue the construction in the preceding manner. The map R in Fig. 1 is replaced by another one.

(a) $p_5 \geqslant 2$. In the starting map (see full lines in Fig. 7) there is again a b-part, and an a-part consisting of two octagonal openings, two pentagons and quadrangles. Cutting off the required k-gons, $k \geqslant 5$, is slightly different from case I because each opening is surrounded by two belts consisting of quadrangles. The cutting paths join always two points on sides of an opening. As an example in Fig. 7 an octagon is cut off from the b-part by dot-and-dash lines.

Balancing the number of vertices on the borders, if needed, and adding the vertex-part to the face-part is performed as in case I. But increasing the number of vertices of an opening, for example O_1, is done in another way. A path joining two points of two edges of the opening O_1 is depicted around the pentagons and the opening O_2 (dashed lines in Fig. 7) forming a new series of quadrangles.

(b) $p_6 \geqslant 1$. The vertices M_1, M_2 as well as the edge $M_1 M_2$ from subcase (a) (Fig. 7) are deleted. The remaining part of the construction is the same as in (a).

Case III. $\sum\limits_{k\geqslant 5}(k-4)p_k = 5, 6, 7, 8$ or 9, $v_5 + 2v_6 + 2v_7 \geqslant 2$, $\sum\limits_{k\geqslant 5}(k-4)v_k < 4$

In fact, there cannot exist a pair of sequences p, v satisfying (1) and $\sum_{k\geqslant 5}(k-4)p_k = 5$ and the other two conditions above. Notice that the last condition implies $v_6 \leqslant 1$, $v_7 \leqslant 1$. Three subcases must be distinguished.

(a) $v_5 \geqslant 2$. In the map in Fig. 7 the vertices M_3 and M_4 are joined by a path in the a-part changing the pentagons into quadrangles. (The double triangle occurs there only if $\sum_{k\geqslant 5}(k-4)p_k \geqslant 8$.) The construction continues as in case II.

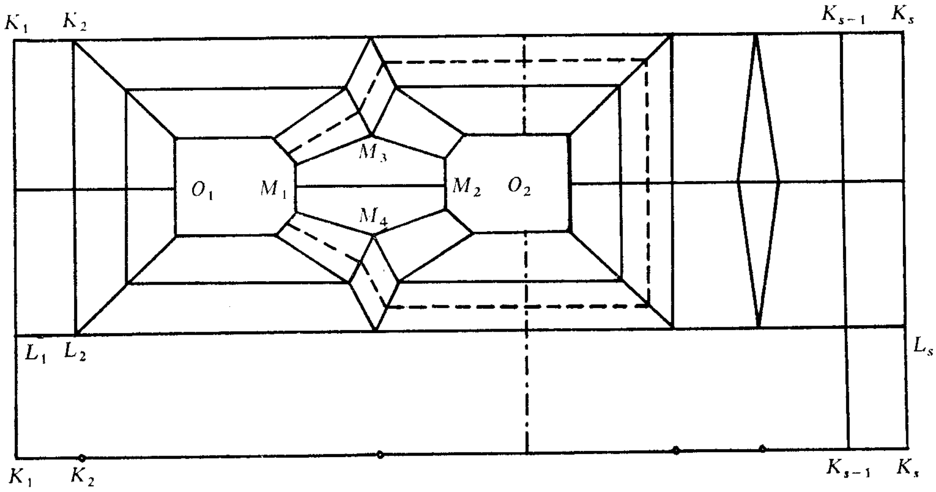

FIG. 7

(b) $v_6 = 1$. The starting map for constructing the face-part is shown in Fig. 8a (full lines) where a vertex V of degree 6 occurs. (In fact, this graph is a small modification of the graph in Fig. 7.) The same procedure as in subcase II(a) (indicated by dashed lines) is used to increase the number of vertices of an opening. Cutting off the required k-gons, $k \geqslant 5$, and adding the vertex-part which can contain a 5-valent and a 3-valent vertex (see Fig. 6), is performed as in that subcase too. (Again in Fig. 8a the double triangle occurs only if $\sum_{k \geqslant 5} (k - 4) p_k \geqslant 8$.)

(c) $v_7 = 1$. From the conditions it follows that $v_3 \geqslant 1$, and $\sum v_i = 0$ for $i \neq 7, 3$. The starting map for cutting off all the required k-gons, $k \geqslant 5$, is depicted in Fig. 8a in full lines supplemented by new edges (dashed lines in Fig. 8b) forming from V a 7-valent vertex and adding a new 3-valent vertex W. Balancing the number of vertices on the openings is performed as in subcase (b).

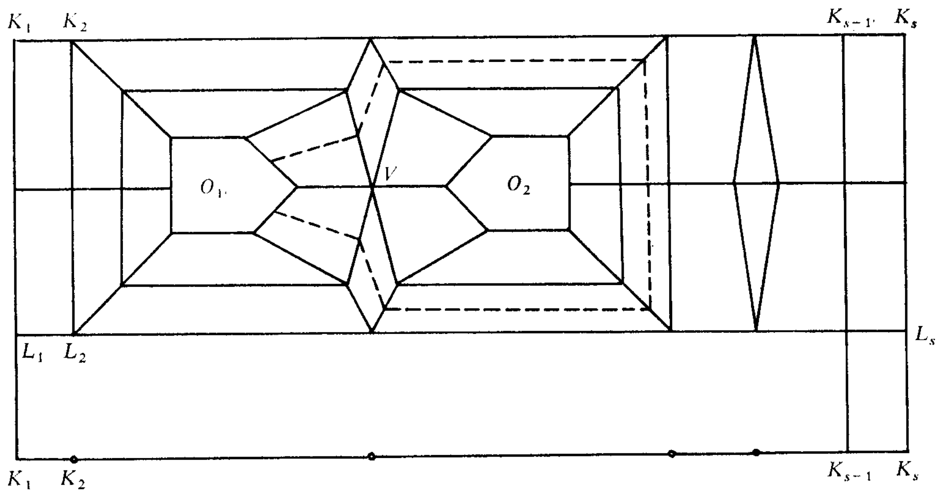

FIG. 8a

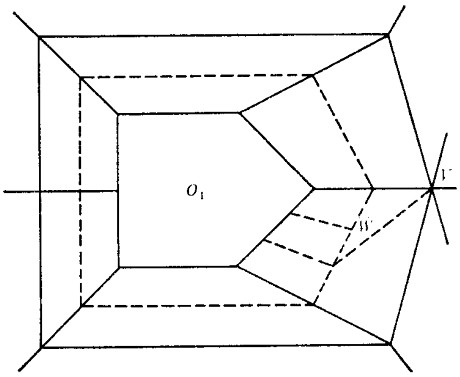

FIG. 8b

Case IV. $\sum_{k\geq 5}(k-4)p_k = 5, 6, 7, 8$ or 9, $p_5 + 2p_6 < 2$, $v_5 + 2v_6 + 2v_7 < 2$, $\sum_{k\geq 5}(k-4)v_k < 4$.

The sequence v can be either $v^0 = (0, 0, \ldots)$ or $v^1 = (1, 1, 0, \ldots)$. It thus follows that there are no pairs p, v of sequences satisfying (1) and the above conditions with $\sum_{k\geq 5}(k-4)p_k = 5, 6$ or 7.

For the sequence p satisfying (1) only one of the following possibilities can occur:

$p^1 \mid p_{12} = 1, p_i = 0$ for all $i \neq 12$;

$p^2 \mid p_7 = p_9 = 1, p_i = 0$ for all $i \neq 7, 9$;

$p^3 \mid p_8 = 2, p_i = 0$ for all $i \neq 8$;

$p^4 \mid p_3 = 1, p_7 = 3, p_i = 0$ for all $i \neq 3, 7$;

$p^5 \mid p_5 = p_{11} = 1, p_i = 0$ for all $i \neq 5, 11$;

$p^6 \mid p_3 = p_5 = p_7 = p_9 = 1, p_i = 0$ for all $i \neq 3, 5, 7, 9$;

$p^7 \mid p_3 = p_5 = 1, p_8 = 2, p_i = 0$ for all $i \neq 3, 5, 8$;

$p^8 \mid p_3 = p_8 = p_9 = 1, p_i = 0$ for all $i \neq 3, 8, 9$;

$p^9 \mid p_3 = p_5 = p_{12} = 1, p_i = 0$ for all $i \neq 3, 5, 12$:

$p^{10} \mid p_3 = p_{13} = 1, p_i = 0$ for all $i \neq 3, 13$;

$p^{11} \mid p_5 = p_7 = p_8 = 1, p_i = 0$ for all $i \neq 5, 7, 8$;

$p^{12} \mid p_3 = p_{10} = p_7 = 1, p_i = 0$ for all $i \neq 3, 7, 10$.

Realizations of all pairs (p^i, v^j), $i = 1, \ldots, 12$, $j = 0, 1$, will now be demonstrated (see Fig. 9).

In Fig. 9a, 9b, 9c and 9d planar maps R_1, R_2, R_3, R_4 are depicted in each of which the pairs of openings a_i, b_i, $i = 1, 2$, should be joined by two handles. From this, realization of the pair (v^0, p^1), (v^0, p^2), (v^0, p^3) and (v^0, p^4), respectively, result. All other realizations (v^0, p^i), $i = 5, \ldots, 12$, are obtained by adding the dashed arcs A

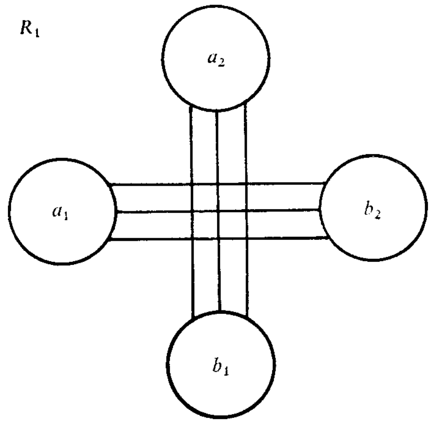

FIG. 9a

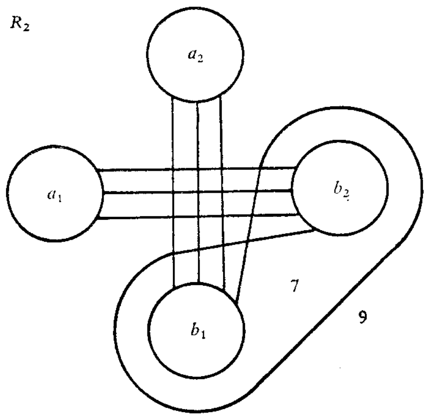

FIG. 9b

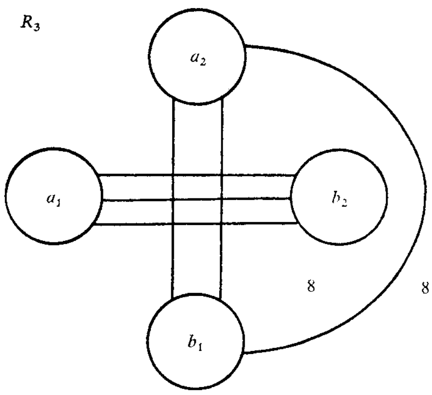

FIG. 9c

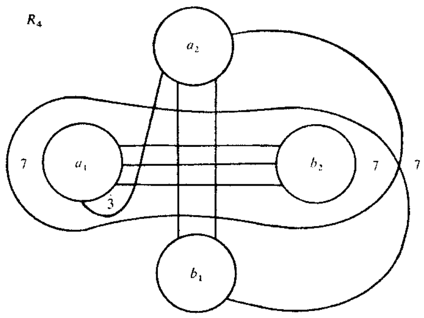

FIG. 9d

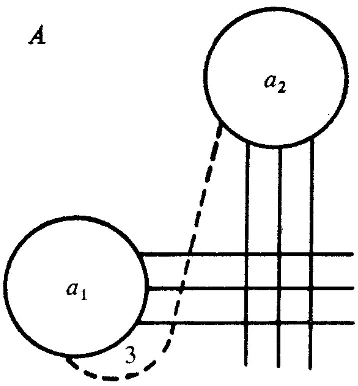

FIG. 9e

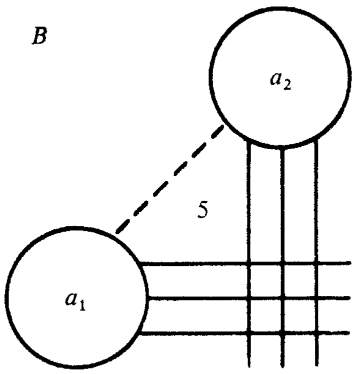

FIG. 9f

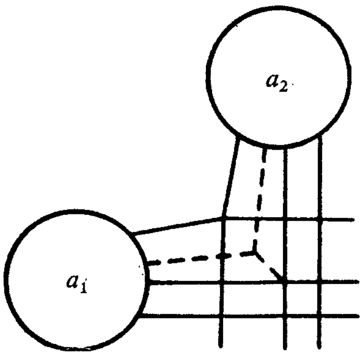

FIG. 9g

or B in Fig. 9e or 9f, respectively (where submaps of R_1, R_2, R_3, R_4 are also depicted) as follows:

(v^0, p^5) results from R_1 and B;

(v_0, p^6) results from R_2 and A and B;

(v_0, p^7) results from R_3 and A and B;

(v^0, p^8) results from R_3 and A (but instead of the openings a_1, a_2 the openings b_1, b_2 are used);

(v^0, p^9) results from R_1 and A and B;

(v^0, p^{10}) results from R_1 and A;

(v^0, p^{11}) results from R_3 and A;

(v^0, p^{12}) results from R_2 and A.

The realization of the pair (v^1, p^i), $i = 1, \ldots, 12$ is obtained from the realization of (v^0, p^i) in adding to it five edges as depicted in Fig. 9g.

Remark. The realizations of the pairs of sequences (v^0, p^5), (v^0, p^{10}) and (v^1, p^1) as well as their duals were kindly communicated by J. Zaks to the authors, who were not able to decide the question of their realizability. Zaks's maps enabled the authors to simplify their original proof of case IV.

Case V. $\sum_{k \geqslant 5} (k-4) p_k \geqslant 4$, $\sum_{k \geqslant 5} (k-4) v_k \geqslant 4$. Two toroidal maps ρ_1, ρ_2 will be constructed as in case I, but containing only one opening each. On ρ_1 whose b-part is a $2[\frac{1}{2} \sum_{k \geqslant 5} (k-4) p_k]$-gon, the k-gonal faces, $k \geqslant 5$, are cut off as before. On ρ_2, whose b-part is a $2[\frac{1}{2} \sum_{k \geqslant 5} (k-4) v_k]$-gon, first v_k k-gonal faces are cut off and the numbers of vertices of the openings are balanced. This last operation is performed so that the opening is joined with the borders of the appropriate ρ_i by the suitable number of paths (dashed lines in Fig. 10a, where four vertices were added to O_i). Then ρ_2 is dualized to ρ_2', and associated with the opening O_2 of ρ_2 is a vertex of ρ_2' incident with triangles only. This vertex is cut off to get an opening O_2' surrounded by quadrangles (Fig. 10b). The maps ρ_1, ρ_2' are joined by identifying their openings.

Case VI. $\sum_{k \geqslant 5} (k-4) p_k < 4$. This case is settled by dualization from cases I–V. This completes the proof of Theorem 1.

3. THEOREM 2. *Every pair of sequences p, v satisfying condition (1) with $g \geqslant 3$ is realizable on P_g.*

Proof. Two cases and a number of subcases must be considered separately. The cases are distinguished depending on the sequence p, without explicitly mentioning in all subcases the properties of the vector v. This should be understood so that *all* vectors v are considered satisfying (1) and the condition of the appropriate subcase. In that way all pairs p, v satisfying (1) are indeed investigated.

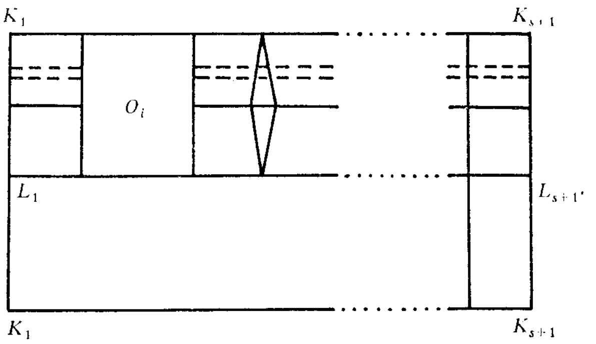

FIG. 10a

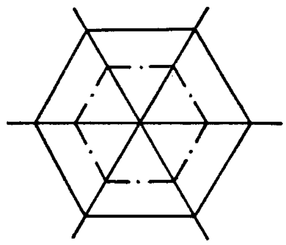

FIG. 10b

Case I. $\sum_{k \geqslant 5} (k-4) p_k \geqslant 8(g-1)$. Analogously as in the proof of Theorem 1 a toroidal map is first constructed. This consists of a $2s$-gon,

$$s = \left[\frac{1}{2} \sum_{k \geqslant 5} (k-4) p_k\right] + \left[\frac{1}{2} \sum_{k \geqslant 5} (k-4) v_k\right],$$

$2(g-1)$ hexagonal openings $O_1, \ldots, O_{2g-2}$, $s-4(g-1)$ double triangles, and quadrangles. (Fig. 11. Notice that the double triangle between O_1 and O_2 is lacking here.) Cutting off the required k-gons, $k \geqslant 5$, forming the dual of the vertex-part, its dualizing and joining with the face-part are performed as in case I of the proof of Theorem 1. However, balancing the number of vertices on the openings and on the borders $K_1 L_1$, $K_{s+1} L_{s+1}$ must be done as follows (*cf.* [4]). In any step when the number of vertices of one opening O_i is increased by two, the numbers of vertices of all openings $O_j \neq O_i$ are also increased by two. This is done by joining pairs of openings to the right and to the left from O_i by two edges (or paths) each. (Fig. 12.) Because on one side of O_i there is an odd number of openings, the last of them is joined by two edges (or paths) with the border. (It can happen that these paths meet the triangles as in Fig. 12.) In order to finish forming the toroidal map the numbers of vertices on the borders $K_1 L_1$, $K_{s+1} L_{s+1}$ must be balanced.

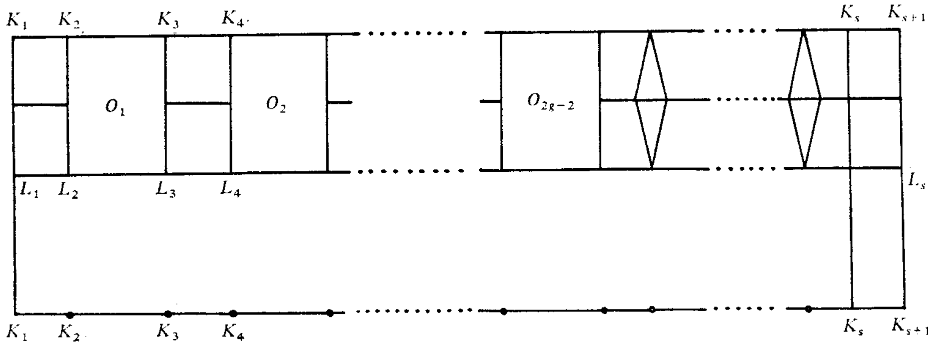

FIG. 11

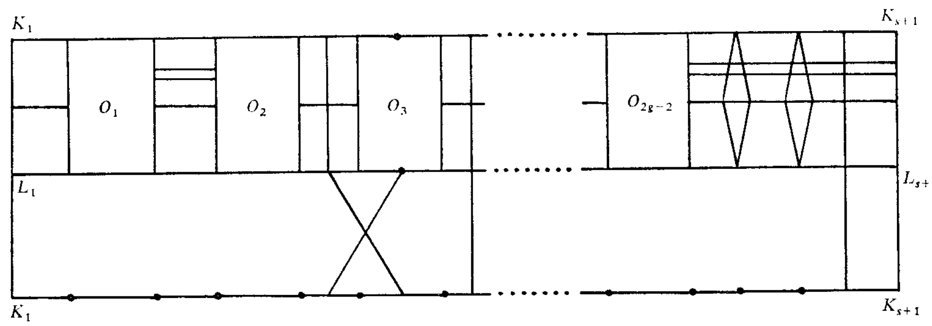

FIG. 12

Suppose, for example, that $r \geqslant y$, where r and y are the number of vertices on the border $K_1 L_1$ and on the border $K_{s+1} L_{s+1}$, respectively.

We choose $r - y$ points $H_1, \ldots, H_{r-y}$ between K_1 and L_1, and $r - y$ points $G_1, \ldots, G_{r-y}$ between K_2 and L_2, and join H_i and G_i by non-intersecting edges (or paths) (Fig. 13). As $g \geqslant 3$ there are at least three openings different from O_1. We choose between the vertices K_5 and L_5 (K_6 and L_6, K_7 and L_7, K_8 and L_8) $c = \frac{1}{2}(r - y)$ points $F_1, \ldots, F_c$ ($N_1, \ldots, N_c$, $P_1, \ldots, P_c$, and $S_1, \ldots, S_c$, respectively), and join F_i with N_i, and P_i with S_i, $i = 1, \ldots, c$, by non-intersecting paths, increasing the number of 3-valent vertices of the openings O_2 and O_4 or O_3 by c or $r - y$, respectively. New quadrangles and possibly 4-valent vertices are also created. At the end of this operation the borders $K_1 L_1$ and $K_{s+1} L_{s+1}$, the pairs of openings O_1 and O_3, O_2 and O_4 have equal numbers of vertices. Of course all other openings $O_5, \ldots, O_{2g-2}$ have equal numbers of vertices too. The last stage of the construction—forming of the $(g - 1)$ handles and adding them to the toroidal map constructed—can now follow.

All the subcases of this case I are precisely identical with those in case I of Theorem I, and are settled in the same manner (except balancing up the numbers of

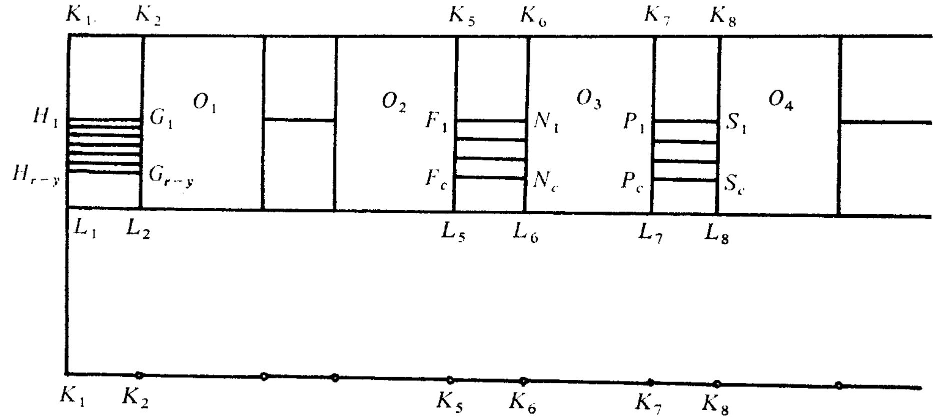

FIG. 13

vertices on the openings and the borders). So, we shall not describe the procedure again.

In case I of Theorem 1, it was assumed that $\sum_{k\geq 5}(k-4)p_k \geq 10(g-1) = 10$, because in the starting map a double triangle was placed between the openings O_1 and O_2 to enable the balancing of numbers of vertices on the openings. However, in the proof of the Theorem 2 this is done in a different way. Notice that in case I of both Theorem 1 and 2, the face-part still includes all the openings.

Remarks. 1. The described procedure of balancing the numbers of vertices on the borders cannot be employed for $g = 2$. That is why a different approach for $g = 2$ was necessary.

2. Notice that in case I as well as in the following case II the increase in the number of vertices of an opening or a border is always even.

Case II. $\sum_{k\geq 5}(k-4)p_k = 8(u-1)+t$, *where* $u = 1, 2, 3, \ldots, g-1$.

Four subcases must be considered, according to the value of the number t.

(a) $t = 0$ or 4. The face-part is constructed as in case I. The assumptions ensure that the arc cutting off the last k-gon, $k \geq 5$, does not meet an opening, and that the face-part does not include all the openings. That is why after forming all k-gons, $k \geq 5$, from the polygon $K_1 K_2 \ldots K_s \ldots L_2 L_1$ v_i i-gons for all $i \geq 5$ are cut off. This will be referred to as the dual of the vertex-part too (although it is not precisely the dual map to that containing the required vertices).

Balancing the number of vertices on the openings and the borders is performed as before. The dual of the vertex-part is separated from the face-part by a series of quadrangles which we include in the dual of the vertex-part, and then this part R_v

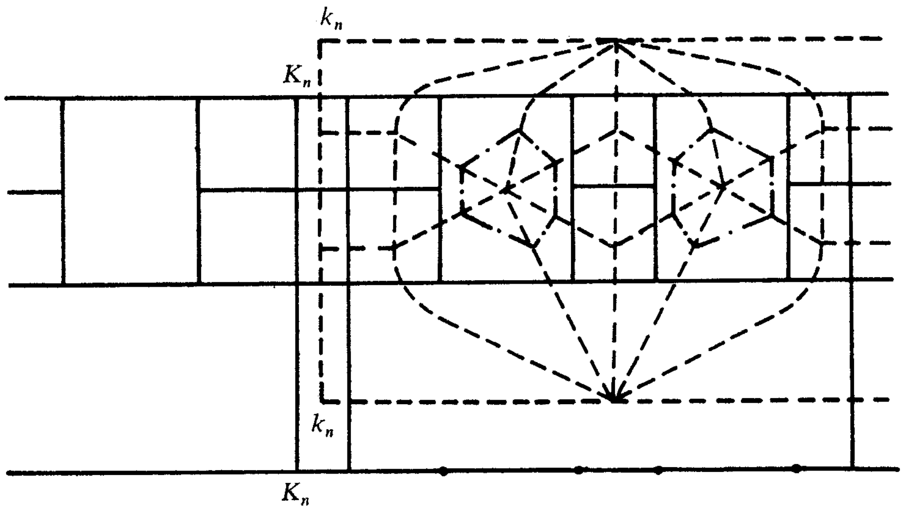

FIG. 14

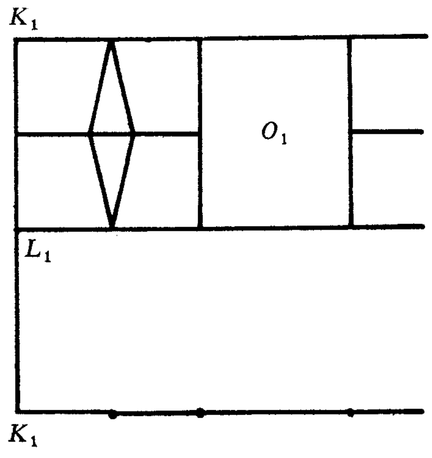

FIG. 15

of the map R is dualized (Fig. 14) analogously as was done in subcase I(b) of Theorem 1. Analogously as in case V of the proof of Theorem 1 the vertices of the vertex-part which are associated to the openings of the map R are cut off to obtain polygons with 3-valent vertices as openings (dot-and-dash lines in Fig. 14) of the vertex-part, and quadrangles. Both the face-part and the vertex-part are then joined as before; handle forming can follow.

(b) $t = 1$ or 5. This condition means that an odd number of k-gonal faces ($k \geqslant 5$ odd) is required. (From (1) it then follows that $p_3 \equiv 1 \pmod 2$.) Forming the face-part is started as in subcase (a). The last face formed is one of the k-gonal faces, where $k \equiv 1 \pmod 2$. This is done in the following way. A $(k-1)$-gonal face F is cut off (the cutting path does not meet an opening). Around the opening, which is adjacent to the face F, a belt consisting of 7 quadrangles and one triangle is put changing F into a k-gon (Fig. 2a). Forming the dual of the vertex-part is performed as in subcase (a), but first the number of vertices on the openings and on the borders is balanced.

(c) $t = 2$ or 6. If we proceeded in this case as in the preceding ones, the arc cutting off the last face in the face-part would meet an opening, and the rest of the construction would fail. The map employed above for forming the face-part must therefore be changed.

From the assumptions made we see that p_3 is even.

(c_1) If $p_3 > 1$ then a double triangle is placed between the border $K_1 L_1$ and the opening O_1 (Fig. 15). If $v_3 > 1$ and $p_3 = 0$ such a double triangle is changed into a pair of trivalent vertices (*cf.* Fig. 5); the construction continues as in subcase (a).

If $p_3 + v_3 = 0$, then at least one of the following possibilities (c_2) or (c_3) must occur.

(c_2) For some odd numbers q, z, $v_q \neq 0$, $v_z \neq 0$, and $z + q \equiv 2 \pmod 4$.

Let $z = q$. The starting toroidal map is the same as in the preceding subcases II(a), (b). (Fig. 11.) However, before forming the face-part, a z-valent and a q-valent vertex will be formed from L_1 and K_1, respectively; this will be done in the following way (Fig. 16a): Choose a point U_1 and V_1 between the vertices L_3 and L_4, and K_3 and K_4, respectively, and join L_1 with U_1, and K_1 with V_1 by edges. Join U_1 and V_1 by a path in the a-part also. Only quadrangles are cut off. Analogously choose a point U_2 and V_2 between the vertices L_4 and L_5, and K_4 and K_5, respectively. Join

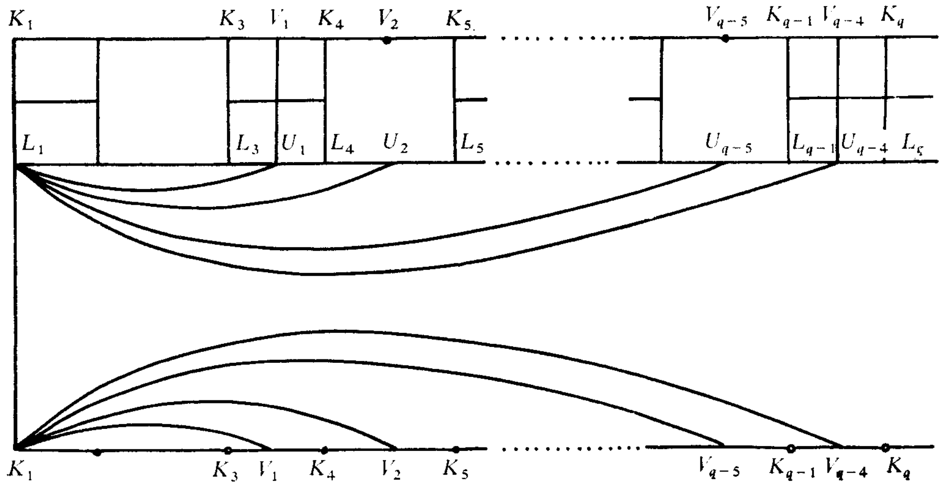

FIG. 16a

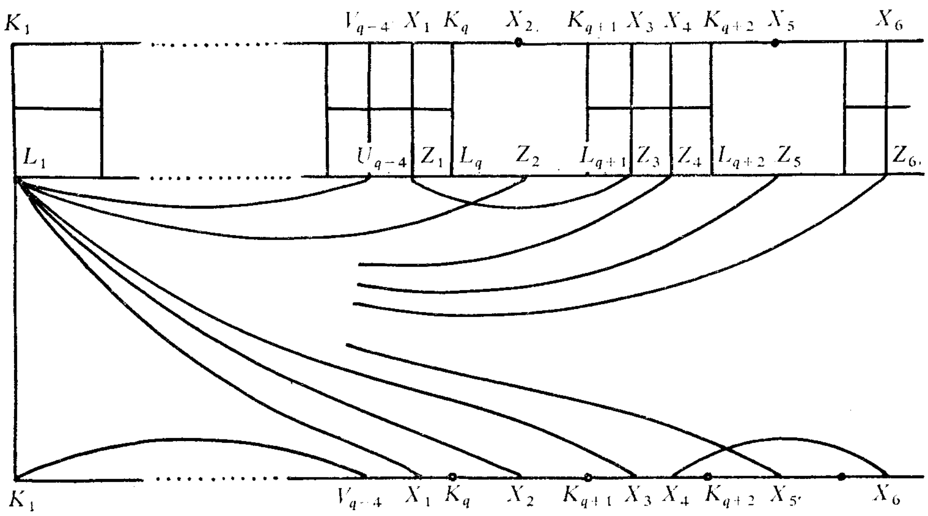

FIG. 16b

L_1 with U_2, and K_1 with V_2 by edges. The vertices U_2, V_2 are not joined by an edge because they are on the boundary of the opening. But balancing the number of vertices of the other openings must be performed at once. This is repeated until K_1 and L_1 have degree $(q-1)$; after identifying the vertices on the borders $K_1 L_1$, $K_{s+1} L_{s+1}$, the vertices K_1, L_1 have degree q.

If $z > q$, then $z - q = 4f$ for some positive integer f. First the vertices K_1 and L_1 are made of degree $q-1$ by the preceding construction. Then the degree of L_1 is increased by 4f (Fig. 16b). A little computation shows that the $(q-4)$-th edge which we have drawn from the vertex L_1 does not meet the boundary of an opening. From the preceding construction it follows that the vertices U_{q-4} and V_{q-4} lie between the vertices L_{q-1} and L_q, and K_{q-1} and K_q, respectively. Join the point L_1 with a point Z_2 lying between L_q and L_{q+1}, with a point X_1 lying between V_{q-4} and K_q, with a point X_2 lying between K_q and K_{q+1}, and with a point X_3 lying between K_{q+1} and K_{q+2}. A path $X_1 Z_1 Z_3 X_3$ is formed where Z_1 and Z_3 lie between U_{q-4} and L_q, and L_{q+1} and L_{q+2}, respectively. So the degree of L_1 is increased by 4, and only new quadrangles and 4-valent vertices have been introduced. If necessary, we now further increase the degree of L_1 by 4. This is done analogously, except that, in a certain sense, the rôles of the vertices Z_i and X_i are interchanged. Join L_1 with a point X_5 lying between K_{q+2} and K_{q+3}, and with a point Z_4 lying between L_{q+3} and L_{q+4}. Form a path $Z_4 X_4 X_6 Z_6$, where X_4 and X_6 lie between X_3 and K_{q+2}, and K_{q+3} and K_{q+4}, respectively. Proceed in this way until L_1 has degree z. In the b-part of the map R there now occurs a $(2s + 8 - z - q)$-gon. From this polygon all required k-gons, $k \geqslant 5$, are cut off and the remaining required vertices are constructed as in subcase (a).

(c_3) For some $k \equiv 2 \pmod 4$, $v_k \neq 0$.

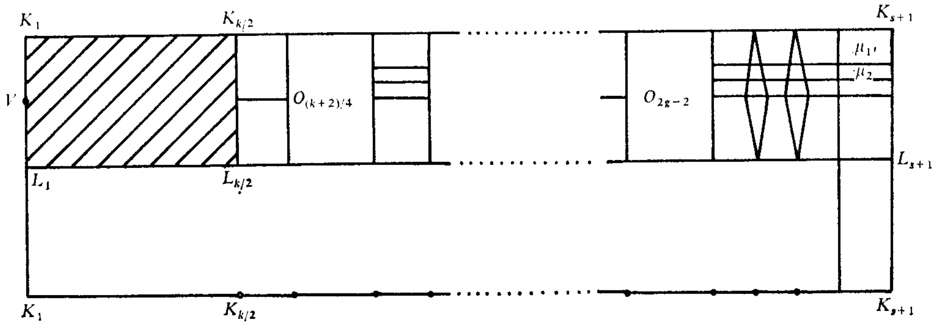

FIG. 17a

In this case the starting map R must yet again be changed, as shown in Fig. 17. This requires an explanation. The graph in Fig. 17b is placed in the cross-hatched area of the graph in Fig. 17a. (If $k \equiv 2 \pmod 8$) the two paths μ_1 and μ_2 do not occur there.) In the map there are a $(k-1)$-valent vertex V, and $\frac{1}{4}(k-2)$ openings $O_1, \ldots, O_{(k-2)/4}$. All openings are octagonal, except O_1 which is a quadrangle. Two vertices of each of the openings $O_2, \ldots, O_{(k-2)/4}$ are joined with V by an edge. The opening O_1 is also joined with V by an edge, and further two belts consisting of quadrangles surrounding O_1 and all $O_2, \ldots, O_{(k-2)/4}$ are inserted. The vertex V is now of degree $k-1$. So a map is obtained in which formation of the face-part and the dualized vertex-part (except one vertex of degree k) can be performed as already described. But some additional explanation is needed concerning balancing the numbers of vertices on the openings and on the borders.

If in a step of the construction the number of vertices of an opening O_i is increased by two, the same is done as above in case I (Fig. 12) with all other openings, with the

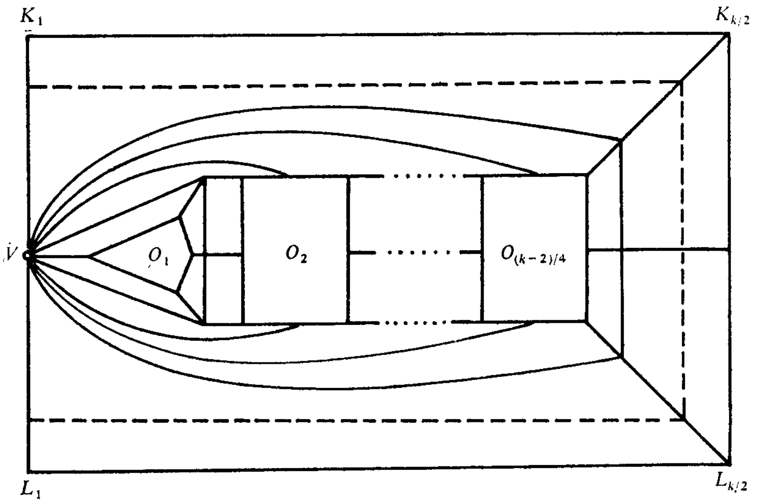

FIG. 17b

possible exception of O_1. (Pairs of openings to the right and to the left from O_i are joined by pairs of paths.) The opening O_1 cannot be joined by paths with the border $K_1 L_1$. In this way all the openings $O_2, \ldots, O_{2g-2}$ have, after forming the face-part and the dual of the vertex-part, the same number r of vertices. Let $y < r$ be the number of vertices of the opening O_1. Join the openings O_1 and O_2 by $(r - y)$ mutually non-intersecting paths; do the same with the opening O_{2g-2} and the border $K_{s+1} L_{s+1}$. After this operation the two openings O_2 and O_{2g-2} have the same number $(2r - y)$ of vertices and can be joined by a handle. All other openings have r vertices. It remains to balance the number of vertices of the borders, i.e. to add an even number of vertices to the border $K_1 L_1$. This is done as depicted in Fig. 17b by dashed lines. Mutually non-intersecting paths join points of the edge $K_1 V$ with points of the edge $V L_1$ forming quadrangles only.

(c_4) p_3 or $v_3 = 1$. For example, we consider the case $v_3 = 1$. From the assumptions made it follows that an odd number n of odd-valent vertices of degree $\geqslant 5$ is required. If $n \geqslant 3$ at least two of them satisfy condition (c_2). If $n = 1$, so that $v_d = 1$ for some odd d, the sequence v is replaced by the sequence v' defined as follows: $v_3' = 0$, $v'_{d-1} = v_{d-1} + 1$, $v_d = 0$, $v_i' = v_i$ for all $i \neq 3, d-1, d$. The sequences p, v' satisfying condition (c_3) are realized, and the vertex V (Fig. 17) is $(d-1)$-valent. The degree of V is increased by one and a trivalent vertex is created as in case IIIc) of the proof of Theorem 1 (Fig. 8b).

(d) $t = 3$ or 7. From this condition it follows that an odd number of k-gonal faces ($k \geqslant 5$ odd) is required, so that $p_3 \geqslant 1$. Suppose, for example, that $j \geqslant 5$ and p_j are both odd numbers. Consider the sequence p' defined as follows: $p_3' = p_3 - 1$, $p'_{j-1} = p_{j-1} + 1$, $p_j' = p_j - 1$, $p_i' = p_i$ for all $i \neq 3, j-1, j$. The sequences p', v satisfy the conditions of subcase (c) and are realized as described above with the following additional demand: the last polygon cut off from the face-part is a $(j-1)$-gon J. As we already know, the last cutting arc does not meet an opening. Let O_i be the last opening contained in the face-part (so it must be adjacent to the polygon J). Arrange this opening as it was done in subcase (b) (Fig. 2a). Then the operations of balancing the number of vertices of the remaining openings and the borders, dualizing the vertex-part, joining the vertex-part with the face-part, etc., can follow.

This completes the proof of Theorem 2, and so of the Main Theorem also.

5. *Remark.* With very small modifications the procedure employed in proofs of our Theorem 1 and 2 can be used in the proof of the Main Theorem for $g = 0$ and 1(cf. [1]). Thus the proof of the whole Main Theorem could be put in a unified form.

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